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Expansions of Cumulative Distribution Functions Directly From Characteristic Functions

Albert H. Nuttall
Surface ASW Directorate

Naval Undersea Warfare Center Detachment
New London, Connecticut

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PREFACE

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13. ABSTRACT *(Maximum 200 words)*
Expansions of probability density functions and their corresponding cumulative distribution functions, directly from given characteristic functions, have been achieved in a series form, with four parameters which are under the control of the user, thereby enabling realization of more rapidly convergent terms. Typical examples of characteristic functions encountered in detection of multiple matched filter outputs are presented, along with an application to a noisy channel with partially-correlated fading. The series presented here enable immediate evaluation of performance of a related normalizer once the expansion coefficients have been determined. An extension to allow for additive Gaussian noise is realized through a set of simple but efficient recursions.

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LIST OF SYMBOLS

$p(u)$	probability density function, (1)
$f(\xi)$	characteristic function, (2)
ν	asymptotic decay, (2), (8)
β	scaling constant, (2)
a_n	expansion coefficient, (2)
z	linear fractional transformation, (5)
α, γ	parameters of transformation, (5)
$\underline{f}(z)$	auxiliary function, (9)
f_n	expansion coefficient of product function, (10)
r	constant $(\beta\alpha - \gamma)/\gamma$, (15)
$S_n(x,\nu)$	auxiliary function, (20)
${}_1F_1$	confluent hypergeometric function, (20)
L_n	Laguerre polynomial, (20)
$C(v)$	cumulative distribution function, (28)
$g(\xi)$	Fourier transform of $C(v)$, (29)
ξ_i	imaginary part of ξ, (29)
$\underline{g}(z)$	auxiliary function, (31)
$G(z)$	product function, (32)
g_n	expansion coefficient of $G(z)$, (32)
$g_o(\xi)$	zeroth order term, (41)
$C_o(v)$	zeroth order distribution, (43)
$\gamma(\nu,x)$	incomplete gamma function, (43)
$g_1(\xi)$	remainder of $g(\xi)$, (44)
$C_1(v)$	remainder of distribution, (44)
$F_n(\xi)$	characteristic function component, (49)

$P_n(u)$	probability density function component, (51)
$C_n(v)$	cumulative distribution function component, (53)
$\underline{C}(\nu,x)$	normalized cumulative distribution function, (54)
$\chi_n(k)$	k-th cumulant of n-th component, (55)
a,ν	type A parameters, (63)
b_m,ν_m	type B parameters, (64)
ϕ_m,c_m	type C parameters, (65)
θ_m,d_m	type D parameters, (66)
t,h	auxiliary constants, (69)
e_n	expansion coefficient of ln F(z), (70)
t,h_m	auxiliary constants, (77)
μ_n	n-th constant, (79)
q_m,h_m	auxiliary constants, (85), (89)
K	number of pulses, (93)
M	number of fading components, (93)
Q_{mk}	random signal-to-noise ratio, (93)
e_{mk}	deterministic signal-to-noise ratio, (93)
ζ	$(1 - \beta/2)/\alpha$, (110)
$E(v)$	exceedance distribution function, (115)
Pr	probability of event, (116)
$p_o(u)$	probability density function of noise-only, (117)
σ	noise level in reference bins, (117)
h	reference random variable, (118)
T,U	auxiliary functions, (120)
σ	additive Gaussian noise standard deviation, (122)
$I_\mu(a)$	auxiliary function, (124)
$J_\mu(a,b)$	auxiliary function, (126)

EXPANSIONS OF CUMULATIVE DISTRIBUTION FUNCTIONS DIRECTLY FROM CHARACTERISTIC FUNCTIONS

INTRODUCTION

The performance analysis of nonlinear signal processors in noise can often be accomplished in closed form in terms of the characteristic function of the system output decision variable; for example, see [1 - 4]. However, due to analytical complexity, determination of the corresponding probability density function and the cumulative or exceedance distribution functions then generally requires some sort of numerical procedure, such as a fast Fourier transform; for example, see [5].

Alternatively, expansions of the probability density function or cumulative distribution function in a Hermite or generalized Laguerre series is possible if the system output high-order moments or cumulants can be easily determined [1,6]. However, this approach of breaking a given characteristic function down into its moments or cumulants, followed by a build-up of a series expansion from coefficients determined recursively from the moments or cumulants [6], can involve a large amount of number-crunching with its attendant round-off error. What is desirable, instead, is a method of proceeding directly from the closed form characteristic function to series expansions for the probability density function and cumulative distribution function. A step in this direction was taken recently in [3,4], where the receiver output characteristic function, for a multiple-pulse processor in

a noisy medium with correlated-fading, was expanded in a series involving only positive terms and coefficients. Here, we will generalize those results and obtain an extension of the generalized Laguerre expansion.

The ideal goals of this analysis would be as follows: the new series expansions should be rapidly convergent and involve as few terms as possible; the series should contain some adjustable parameters that can be modified to achieve more rapid convergence, especially for difficult situations such as very low false-alarm probabilities or very high signal-to-noise ratios; the individual terms in the series should be easily computed, preferably by efficient recursions; the series for the probability density function should be easily integrable to yield the cumulative or exceedance distribution functions; and the series should involve only positive terms, if feasible, in order to retain high accuracy in the tail probability regions.

The following results have achieved partial success in realizing these goals. The expansions are not limited to orthogonal expansions, such as the Laguerre and Hermite cases. There exists the possibility of creating additional convergence factors in the expansions which speed convergence significantly. And, arbitrary signal-to-noise ratios and thresholds can be easily accommodated without sacrificing accuracy. There still exists the problem of deciding how best to pick the free parameters in the expansions for most rapid convergence. It is generally not simply to make a few low-order coefficients zero; this is consistent with [6; page 3 and examples].

EXPANSION OF CHARACTERISTIC FUNCTION

Before presenting the new expansions, it is worthwhile to review some earlier results. The generalized Laguerre expansions of a probability density function p(u), for a positive random variable, and its corresponding characteristic function (Fourier transform) $f(\xi)$ are given, respectively, by [6; (102)] as

$$p(u) = \frac{u^{\nu-1}\exp(-u/\beta)}{\beta^{\nu}\,\Gamma(\nu)} \sum_{n=0}^{\infty} a_n\, L_n^{(\nu-1)}\left(\frac{u}{\beta}\right) \quad \text{for } u > 0 , \tag{1}$$

$$f(\xi) = \frac{1}{(1 - i\xi\beta)^{\nu}} \sum_{n=0}^{\infty} a_n \frac{(\nu)_n}{n!} \left(\frac{-i\xi\beta}{1 - i\xi\beta}\right)^n . \tag{2}$$

Except for the leading factor $1/(1 - i\xi\beta)^{\nu}$, the basic idea here can be interpreted as the expansion of the characteristic function $f(\xi)$ in a power series of the quantity $-i\xi/(1 - i\xi\beta)$. There are two free parameters in (2), namely ν and β, which can be chosen to speed up convergence, that is, hasten the decay of the terms in (1) and (2). Observe that $a_n = a_n(\nu,\beta)$.

An alternative approach is given by the following expansion of the characteristic function, involving only powers of the factor $1/(1 - i\xi\beta)$; see [3; appendix D] and [4; appendix A]:

$$p(u) = \frac{u^{\nu-1}\exp(-u/\beta)}{\beta^{\nu}\,\Gamma(\nu)} \sum_{n=0}^{\infty} a_n \frac{(u/\beta)^n}{(\nu)_n} \quad \text{for } u > 0 , \tag{3}$$

$$f(\xi) = \frac{1}{(1 - i\xi\beta)^{\nu}} \sum_{n=0}^{\infty} a_n \left(\frac{1}{1 - i\xi\beta}\right)^n . \tag{4}$$

Again, there are two free parameters, ν and β, and $a_n = a_n(\nu,\beta)$.

LINEAR FRACTIONAL TRANSFORMATION

Both of the expansions in (2) and (4), above, are special cases of a linear fractional (or bilinear) transformation

$$z = \frac{\alpha - i\xi\gamma}{1 - i\xi\beta} , \quad i\xi = \frac{\alpha - z}{\gamma - \beta z} , \quad \gamma \neq \beta\alpha , \quad \beta > 0 , \tag{5}$$

between the complex ξ and z planes. A linear fractional transformation is the most general analytic transformation yielding a one-to-one correspondence between all points in the simple ξ and z planes [7; page 365]. (The transformation in (2) corresponds to $\alpha = 0$, $\gamma = \beta$, while that in (4) utilizes $\gamma = 0$, $\alpha = 1$.)

We observe the following associations:

$$\xi = 0 \text{ maps into } z = \alpha ;$$

$$\xi = \varepsilon << 1 \text{ maps into } z \sim (\alpha - i\varepsilon\gamma)(1 + i\varepsilon\beta) \sim \alpha + i\varepsilon(\beta\alpha - \gamma) ;$$

$$\xi = \infty \text{ maps into } z = \gamma/\beta . \tag{6}$$

Then, for positive parameters α,β,γ, the upper half ξ-plane maps into the interior of a circle in the z-plane, with center c and radius r', where

$$c = \frac{1}{2}\left(\alpha + \frac{\gamma}{\beta}\right) , \quad r' = \frac{1}{2}\left|\alpha - \frac{\gamma}{\beta}\right| . \tag{7}$$

This result holds regardless of whether $\beta\alpha > \gamma$ or $\beta\alpha < \gamma$.

EXPANSION PROCEDURE

To demonstrate how the expansions are developed, suppose characteristic function $f(\xi)$ is given, and that its asymptotic behavior is governed by

$$f(\xi) \sim c/(i\xi)^{\nu} \quad \text{as } \xi \to \infty , \quad \nu > 0 . \tag{8}$$

This behavior determines parameter ν. Now, define auxiliary function

$$\underline{f}(z) = f\left(\xi = \frac{1}{i}\,\frac{\alpha - z}{\gamma - \beta z}\right) . \tag{9}$$

Form the following product, and then develop the product in a power series expansion in z according to

$$\left(\frac{\gamma - \beta\alpha}{\gamma - \beta z}\right)^{\nu} \underline{f}(z) = \sum_{n=0}^{\infty} f_n\, z^n \; ; \qquad f_n = f_n(\nu,\beta,\alpha,\gamma) . \tag{10}$$

Then, from (9) and (5), the original characteristic function is

$$f(\xi) = \underline{f}\left(z = \frac{\alpha - i\xi\gamma}{1 - i\xi\beta}\right) = \frac{1}{(1 - i\xi\beta)^{\nu}} \sum_{n=0}^{\infty} f_n \left(\frac{\alpha - i\xi\gamma}{1 - i\xi\beta}\right)^n . \tag{11}$$

The scale factor $\gamma - \beta\alpha$ is used in the product on the left side of (10) in order to get the more compact form (11). The asymptotic behavior of expansion (11) agrees with (8), due to the introduction of leading factor $(\gamma - \beta\alpha)/(\gamma - \beta z) = 1 - i\xi\beta$ in (10), <u>before</u> the power series expansion in z is accomplished.

The coefficients $\{f_n\}$ in (10) and (11) are functions of the four parameters ν,β,α,γ. If $\alpha = 0$, the quantity $f_n(\nu,\beta,0,\gamma)\ \gamma^n$

in (11) can be denoted as $\tilde{f}_n(\nu,\beta)$, giving form (2). On the other hand, if $\gamma = 0$, the quantity $f_n(\nu,\beta,\alpha,0)\ \alpha^n$ in (11) can be denoted as $\tilde{f}_n(\nu,\beta)$, giving form (4).

Although four parameters are available in (11), only three are fundamental; that may be seen by writing (11) in the form

$$f(\xi) = \frac{1}{(1 - i\xi\beta)^{\nu}} \sum_{n=0}^{\infty} f_n\ \alpha^n \left(\frac{1 - i\xi\gamma/\alpha}{1 - i\xi\beta}\right)^n , \qquad (12)$$

provided that $\alpha \neq 0$. Coefficient $f_n\ \alpha^n$ is then a function only of the three parameters ν, β, α/γ. Since characteristic function value $f(0) = 1$, there follows from (12) the useful constraint

$$\sum_{n=0}^{\infty} f_n\ \alpha^n = 1 . \qquad (13)$$

Another possible triplet of parameters is

ν	asymptotic decay ;
β	scaling constant ;
$\beta\alpha/\gamma$	matching constant . (14)

Whichever set of three parameters are used, they should be chosen so that series (11) or (12) has rapidly decaying coefficients $\{f_n\}$, if possible.

EXPANSION OF PROBABILITY DENSITY FUNCTION

In this section, the probability density function corresponding to characteristic function expansion (11) will be developed. For future use, we define

$$r = \frac{\beta\alpha - \gamma}{\gamma} , \quad \gamma \neq 0 . \tag{15}$$

We begin by observing that the numerator term $(\alpha - i\xi\gamma)^n$ in (11) can be expressed as

$$(\alpha - i\xi\gamma)^n = \left(\frac{\gamma}{\beta}\right)^n \left(\frac{\beta\alpha - \gamma}{\gamma} + 1 - i\xi\beta\right)^n =$$

$$= \left(\frac{\gamma}{\beta}\right)^n \sum_{m=0}^{n} \binom{n}{m} r^m (1 - i\xi\beta)^{n-m} . \tag{16}$$

This enables us to express the n-th term in (11) according to

$$\frac{1}{(1 - i\xi\beta)^\nu} \left(\frac{\alpha - i\xi\gamma}{1 - i\xi\beta}\right)^n = \left(\frac{\gamma}{\beta}\right)^n \sum_{m=0}^{n} \binom{n}{m} r^m (1 - i\xi\beta)^{-\nu-m} . \tag{17}$$

Then, characteristic function (11) takes the desired form

$$f(\xi) = \sum_{n=0}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n \sum_{m=0}^{n} \binom{n}{m} r^m (1 - i\xi\beta)^{-\nu-m} . \tag{18}$$

The expansion of probability density function $p(u)$, corresponding to this form of $f(\xi)$ in (18), is

$$p(u) = \frac{(u/\beta)^{\nu-1} \exp(-u/\beta)}{\beta\, \Gamma(\nu)} \sum_{n=0}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n S_n\left(\frac{ru}{\beta}, \nu\right) \quad \text{for } u > 0 , \tag{19}$$

where auxiliary function

$$S_n(x,\nu) \equiv \sum_{m=0}^{n} \binom{n}{m} \frac{x^m}{(\nu)_m} = {}_1F_1(-n;\nu;-x) = \frac{n!}{(\nu)_n} L_n^{(\nu-1)}(-x) \ . \qquad (20)$$

Here, we used [8; (13.6.9)]. The recursion for $S_n(x,\nu)$ is, by means of [8; (13.4.1)],

$$S_n(x,\nu) = \frac{(\nu+2n-2+x)\ S_{n-1}(x,\nu) - (n-1)\ S_{n-2}(x,\nu)}{\nu+n-1} \qquad (21)$$

for $n \geq 1$, with $S_0(x,\nu) = 1$. Value $S_{-1}(x,\nu)$ can be arbitrarily set to zero, since it gets multiplied by zero when $n = 1$ in (21).

From (15), if $\gamma < \beta\alpha$, then $r > 0$ and all the terms in $S_n(ru/\beta,\nu)$ in (20) are positive. Sequence $\{S_n(ru/\beta,\nu)\}$ in (19) does not decay rapidly to zero with n; in fact, the sequence can grow slowly with n, depending on the polarity of r. Convergence of (19) must be furnished by sequence $f_n\ (\gamma/\beta)^n$.

We could express the probability density function in (19) as

$$p(u) = \frac{(u/\beta)^{\nu-1} \exp(-u/\beta)}{\beta\ \Gamma(\nu)} \sum_{n=0}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n \frac{n!}{(\nu)_n} L_n^{(\nu-1)}\left(\frac{-ru}{\beta}\right) \ . \qquad (22)$$

(Then, by using [9; 7.414 8], the corresponding characteristic function is readily obtained as (11), which constitutes a check on the above developments.) Notice that the argument, $-ru/\beta$, of the Laguerre polynomial L_n in (22) is not the same as the argument, u/β, of the exponential, unless $r = -1$, that is, unless we are dealing with the special case of $\alpha = 0$.

For $\alpha = 0$, the results above reduce to

$$p(u) = \frac{(u/\beta)^{\nu-1} \exp(-u/\beta)}{\beta\ \Gamma(\nu)} \sum_{n=0}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n \frac{n!}{(\nu)_n} L_n^{(\nu-1)}\left(\frac{u}{\beta}\right) \text{ for } u>0 \quad (23)$$

$$f(\xi) = \frac{1}{(1 - i\xi\beta)^{\nu}} \sum_{n=0}^{\infty} f_n \left(\frac{-i\xi\gamma}{1 - i\xi\beta}\right)^n . \quad (24)$$

These agree with the usual generalized Laguerre expansion in [6; (102)] when we set $\gamma = \beta$ and identify ν, f_n here with $\alpha+1$, $a_n\ (\alpha+1)_n/n!$ there, respectively.

On the other hand, in the limit as $\gamma \to 0$, observe from (20) that

$$\left(\frac{\gamma}{\beta}\right)^n S_n\left(\frac{ru}{\beta},\nu\right) = \left(\frac{\gamma}{\beta}\right)^n \sum_{m=0}^{n} \binom{n}{m} \frac{1}{(\nu)_m} \left(\frac{\beta\alpha - \gamma}{\gamma}\frac{u}{\beta}\right)^m =$$

$$\sim \left(\frac{\gamma}{\beta}\right)^n \frac{1}{(\nu)_n} \left(\frac{\beta\alpha}{\gamma}\frac{u}{\beta}\right)^n = \frac{1}{(\nu)_n} \left(\frac{\alpha u}{\beta}\right)^n \text{ as } \gamma \to 0 . \quad (25)$$

Then, the result in (19) reduces (as $\gamma \to 0$) to

$$p(u) = \frac{(u/\beta)^{\nu-1} \exp(-u/\beta)}{\beta\ \Gamma(\nu)} \sum_{n=0}^{\infty} f_n \frac{(\alpha u/\beta)^n}{(\nu)_n} \text{ for } u > 0 , \quad (26)$$

while (11) becomes

$$f(\xi) = \frac{1}{(1 - i\xi\beta)^{\nu}} \sum_{n=0}^{\infty} f_n \left(\frac{\alpha}{1 - i\xi\beta}\right)^n . \quad (27)$$

These agree with [3; (D-15) and (D-13)], respectively, when we set $\alpha = 1$, $\beta = 2$, and identify ν, f_n here with K, F g_n there, respectively.

Relations (22) and (19) are extensions of the generalized Laguerre expansion of a probability density function. These are the desired results, in so far as the probability density function is concerned; we now proceed to expansions for the corresponding cumulative distribution function.

EXPANSION OF CUMULATIVE DISTRIBUTION FUNCTION

The cumulative distribution function is

$$C(v) = \int_{-\infty}^{v} du \; p(u) \; . \tag{28}$$

We will obtain expansions for C(v) in two different ways, each with its own merits and limitations.

FOURIER TRANSFORM OF C(v)

The Fourier transform of the cumulative distribution function C(v) is considered in appendix A; it is given in terms of the characteristic function $f(\xi)$ by

$$g(\xi) = \int dv \, \exp(i\xi v) \, C(v) = \frac{-1}{i\xi} f(\xi) \quad \text{for } 0 < \xi_i < a \; , \tag{29}$$

where ξ_i is the imaginary part of ξ. Constant a is positive, and can be $+\infty$, as for a random variable which is always positive. Then, from (8), the asymptotic behavior of $g(\xi)$ is

$$g(\xi) \sim \frac{-c}{(i\xi)^{\nu+1}} \quad \text{as } \xi \to \infty \; . \tag{30}$$

Analogous to (9) and sequel, we define auxiliary function

$$\underline{g}(z) \equiv g\left(\xi = \frac{1}{i}\,\frac{\alpha - z}{\gamma - \beta z}\right) = -\,\frac{\gamma - \beta z}{\alpha - z}\,\underline{f}(z) \; , \tag{31}$$

where we used (29). Now, we form the following product, G(z), and develop the product in a power series in z according to

$$G(z) \equiv \left(\frac{\gamma - \beta\alpha}{\gamma - \beta z}\right)^{\nu+1} \underline{g}(z) = \sum_{n=0}^{\infty} g_n \; z^n \; ; \quad g_n = g_n(\nu,\beta,\alpha,\gamma) \; . \tag{32}$$

In order to find coefficients $\{g_n\}$, we use (31) and (10) to find

$$G(z) = - \frac{(\gamma - \beta\alpha)^{\nu+1}}{(\gamma - \beta z)^{\nu} \; (\alpha - z)} \underline{f}(z) = \frac{\beta\alpha - \gamma}{\alpha - z} \sum_{n=0}^{\infty} f_n \; z^n \; . \tag{33}$$

Now, if $\alpha \neq 0$, we expand $(\alpha - z)^{-1} = \alpha^{-1} \; (1 - z/\alpha)^{-1}$ in a power series in z, and combine the two series in (33), thereby obtaining coefficients $\{g_n\}$ in (32) as

$$g_n = \frac{\beta\alpha - \gamma}{\alpha^{n+1}} \sum_{m=0}^{n} f_m \; \alpha^m = \frac{1}{\alpha} \; g_{n-1} + \frac{\beta\alpha - \gamma}{\alpha} \; f_n \quad \text{for } n \geq 0 \; , \tag{34}$$

with $g_{-1} = 0$. Thus, coefficients $\{g_n\}$ are readily found in terms of the earlier coefficients $\{f_n\}$ in (10), at least when $\alpha \neq 0$. Finally, using (31) and (32), we have the desired expansion for the Fourier transform (29), namely

$$g(\xi) = \underline{g}\left(z = \frac{\alpha - i\xi\gamma}{1 - i\xi\beta}\right) = \frac{1}{(1 - i\xi\beta)^{\nu+1}} \sum_{n=0}^{\infty} g_n \left(\frac{\alpha - i\xi\gamma}{1 - i\xi\beta}\right)^n \; , \tag{35}$$

where $g_n = g_n(\nu,\beta,\alpha,\gamma)$.

Observe from (34) that

$$\lim_{n\to\infty} \; g_n \; \alpha^{n+1} = (\beta\alpha - \gamma) \sum_{m=0}^{\infty} f_m \; \alpha^m = \beta\alpha - \gamma \; , \tag{36}$$

where we used (13). Therefore, the asymptotic decay of coefficients $\{g_n\}$ is according to

$$g_n \sim \frac{\beta\alpha - \gamma}{\alpha^{n+1}} \quad \text{as } n \to \infty \; . \tag{37}$$

This result demonstrates that series (35) will converge if $|\gamma|$ is less than $|\beta\alpha|$.

Then, by reference to the Fourier transform pair (11) and (19), the cumulative distribution function (inverse Fourier transform) corresponding to (35) is immediately available as

$$C(v) = \frac{(v/\beta)^{\nu} \exp(-v/\beta)}{\beta\, \Gamma(\nu+1)} \sum_{n=0}^{\infty} g_n \left(\frac{\gamma}{\beta}\right)^n S_n\left(\frac{rv}{\beta}, \nu+1\right) =$$

$$= \frac{(v/\beta)^{\nu} \exp(-v/\beta)}{\beta\, \Gamma(\nu+1)} \sum_{n=0}^{\infty} g_n \left(\frac{\gamma}{\beta}\right)^n \frac{n!}{(\nu+1)_n} L_n^{(\nu)}\left(\frac{-rv}{\beta}\right) \quad \text{for } v > 0 \; , \tag{38}$$

where we also used (20). These results apply only for $\alpha \neq 0$. The series converges if $|\gamma| < |\beta\alpha|$, as may be seen by reference to (37) and [10; (8.22.1)]. The relations in (38) are the main results of this subsection.

If $0 < \gamma < \beta\alpha$, then $r > 0$ from (15), and all the terms in $S_n(rv/\beta, \nu+1)$ are positive. Again, notice that the argument, $-rv/\beta$, of the Laguerre polynomial L_n is not the same as the argument, v/β, of the exponential, unless $r = -1$, that is, unless we are dealing with the special case of $\alpha = 0$. However, this special case cannot be obtained from the above by simply letting $\alpha \to 0$ in (38); see the comment under (33), where property $\alpha \neq 0$ was explicitly used in developing the series expansion for $(1 - z/\alpha)^{-1}$.

SPECIAL CASE OF $\alpha = 0$

From (32) and (33), when $\alpha = 0$, we have

$$\underline{g}(z) = \left(\frac{\gamma - \beta z}{\gamma}\right)^{\nu+1} G(z) = \frac{(\gamma - \beta z)^{\nu+1}}{\gamma^{\nu}} \sum_{n=0}^{\infty} f_n \, z^{n-1} \, , \qquad (39)$$

with $f_n = f_n(\nu,\beta,0,\gamma)$. Then, (35) yields

$$g(\xi) = \underline{g}\left(z = \frac{-i\xi\gamma}{1 - i\xi\beta}\right) = \frac{\gamma}{(1 - i\xi\beta)^{\nu+1}} \sum_{n=0}^{\infty} f_n \left(\frac{-i\xi\gamma}{1 - i\xi\beta}\right)^{n-1} . \qquad (40)$$

Again, we could let $f_n(\nu,\beta,0,\gamma)\ \gamma^n = \tilde{f}_n(\nu,\beta)$, where the latter quantity is independent of γ.

The cumulative distribution function term corresponding to the $n = 0$ term in (40) requires special treatment; we have

$$g_o(\xi) \equiv \frac{\gamma}{(1 - i\xi\beta)^{\nu+1}} \, f_o \, \frac{1 - i\xi\beta}{-i\xi\gamma} = \frac{f_o}{-i\xi(1 - i\xi\beta)^{\nu}} \, . \qquad (41)$$

Since we know the Fourier transform correspondence

$$(1 - i\xi\beta)^{-\nu} \longleftrightarrow \frac{u^{\nu-1} \exp(-u/\beta)}{\beta^{\nu} \, \Gamma(\nu)} \quad \text{for } u > 0 \, , \qquad (42)$$

the $g_o(\xi)$ term corresponds to the cumulative distribution function component (see (28) and (29))

$$C_o(v) = f_o \int_0^v du \, \frac{u^{\nu-1} \exp(-u/\beta)}{\beta^{\nu} \, \Gamma(\nu)} = f_o \, \frac{\gamma(\nu, v/\beta)}{\Gamma(\nu)} =$$

$$= f_o \frac{(v/\beta)^\nu \exp(-v/\beta)}{\Gamma(\nu+1)} \; {}_1F_1\left(1;1+\nu;\frac{v}{\beta}\right) \quad \text{for } v > 0 \;, \tag{43}$$

where $\gamma(\nu,x)$ is the incomplete gamma function [8; (6.5.2)], and ${}_1F_1$ is the confluent hypergeometric function [8; (6.5.12)].

The remainder, $g_1(\xi)$, of $g(\xi)$ in (40) can be inverse transformed immediately by comparing Fourier transform pair (11) and (19). Its cumulative distribution function component is

$$C_1(v) = \frac{(v/\beta)^\nu \exp(-v/\beta)}{\Gamma(\nu+1)} \sum_{n=1}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n S_{n-1}\left(\frac{-v}{\beta},\nu+1\right) \tag{44}$$

for $v > 0$. The total cumulative distribution function is given by the sum of (43) and (44):

$$C(v) = C_o(v) + C_1(v) = \tag{45}$$

$$= \frac{(v/\beta)^\nu \exp(-v/\beta)}{\Gamma(\nu+1)} \left[f_o \; {}_1F_1\left(1;1+\nu;\frac{v}{\beta}\right) + \sum_{n=1}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n S_{n-1}\left(\frac{-v}{\beta},\nu+1\right)\right] .$$

This expansion for the cumulative distribution function applies for $\alpha = 0$. Again, we could let $f_n(\nu,\beta,0,\gamma)\;\gamma^n = \tilde{f}_n(\nu,\beta)$.

An alternative expression for $C(v)$ is available when we use identity (20) on result (45). When this is done, the result for $C(v)$ in (45) agrees with [6; (95)], when we set $\gamma = \beta$ and identify ν, f_n here with $\alpha+1$, $a_n\;(\alpha+1)_n/n!$ there, respectively.

SPECIAL CASE OF $\gamma = 0$

As another check on general result (38), we consider letting $\gamma \to 0$. From (25), we have

$$\left(\frac{\gamma}{\beta}\right)^n S_n\left(\frac{rv}{\beta}, \nu+1\right) \sim \frac{1}{(\nu+1)_n} \left(\frac{\alpha v}{\beta}\right)^n \quad \text{as } \gamma \to 0 . \tag{46}$$

Then, from (38), when $\gamma = 0$, the cumulative distribution function becomes

$$C(v) = \frac{(v/\beta)^\nu \exp(-v/\beta)}{\beta\, \Gamma(\nu+1)} \sum_{n=0}^{\infty} g_n \frac{(\alpha v/\beta)^n}{(\nu+1)_n} =$$

$$= \frac{(v/\beta)^\nu \exp(-v/\beta)}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} f_m \alpha^m\right) \frac{(v/\beta)^n}{(\nu+1)_n} \quad \text{for } v > 0 . \tag{47}$$

Here, also, we could let $f_m(\nu,\beta,\alpha,0)\, \alpha^m = \tilde{f}_m(\nu,\beta)$.

The quantity $1 - C(v)$ is a rearrangement of [3; (D-21)]; agreement can be confirmed when we set $\alpha = 1$, $\beta = 2$, use (34), and identify ν, f_n here with K, F g_n there, respectively.

INTEGRATION OF PROBABILITY DENSITY FUNCTION EXPANSION

At the beginning of this section, it was noted that expansions for the cumulative distribution function C(v) would be obtained by two different methods. The results above were obtained by means of the Fourier transform of C(v); now, we integrate directly on the probability density function p(u).

From (12), the characteristic function $f(\xi)$, for arbitrary ν, β, α, γ, can be expressed as

$$f(\xi) = \sum_{n=0}^{\infty} f_n \, \alpha^n \, F_n(\xi) \ , \quad \alpha \neq 0 \ , \tag{48}$$

where characteristic function component

$$F_n(\xi) \equiv \frac{(1 - i\xi\gamma/\alpha)^n}{(1 - i\xi\beta)^{\nu+n}} \ , \quad \nu > 0 \ . \tag{49}$$

Component $F_n(\xi)$ is a characteristic function, in that $F_n(0) = 1$ for all n. Then, from (19), the probability density function corresponding to (48) is

$$p(u) = \sum_{n=0}^{\infty} f_n \, \alpha^n \, P_n(u) \quad \text{for } u > 0 \ , \tag{50}$$

where probability density component

$$P_n(u) \equiv \left(\frac{\gamma}{\beta\alpha}\right)^n \frac{(u/\beta)^{\nu-1} \exp(-u/\beta)}{\beta \, \Gamma(\nu)} \, S_n\left(\frac{ru}{\beta}, \nu\right) \quad \text{for } u > 0 \ . \tag{51}$$

Component $P_n(u)$ is a probability density function, in that $\int du \, P_n(u) = 1$ for all n.

There follows, from (50), the cumulative distribution function

$$C(v) = \int_0^v du\ p(u) = \sum_{n=0}^{\infty} f_n\ \alpha^n\ C_n(v) \quad \text{for } v > 0\ , \tag{52}$$

where cumulative distribution component

$$C_n(v) = \int_0^v du\ P_n(u) = \left(\frac{\gamma}{\beta\alpha}\right)^n \sum_{m=0}^{n} \binom{n}{m}\ r^m \int_0^v \frac{du}{\beta}\ \frac{(u/\beta)^{\nu+m-1}\ \exp(-u/\beta)}{\Gamma(\nu+m)} =$$

$$= \left(\frac{\gamma}{\beta\alpha}\right)^n \sum_{m=0}^{n} \binom{n}{m}\ r^m\ \underline{C}\left(\nu+m, \frac{v}{\beta}\right) \quad \text{for } v > 0\ ;\quad r = \frac{\beta\alpha - \gamma}{\gamma}\ . \tag{53}$$

Here, we have used definition

$$\underline{C}(\nu, x) \equiv \int_0^x dy\ \frac{y^{\nu-1}\ \exp(-y)}{\Gamma(\nu)} \quad \text{for } x \geq 0\ ,\quad \nu > 0\ ; \tag{54}$$

see appendix B for additional useful recurrence relations on this incomplete gamma function. Component $C_n(v)$ is a cumulative distribution function, in that $C_n(\infty) = 1$ (using $\underline{C}(a,\infty) = 1$). Relations (52) - (53) are the main results of this subsection.

The cumulants corresponding to an individual component may be found readily from (49); the k-th order cumulant is

$$\chi_n(k) = (k-1)!\ \left((\nu+n)\ \beta^k - n\ \left(\frac{\gamma}{\alpha}\right)^k\right) \quad \text{for } k \geq 1\ . \tag{55}$$

In particular,

$$\chi_n(1) = (\nu+n)\ \beta - n\ \frac{\gamma}{\alpha}\ ,$$

$$X_n(2) = (\nu+n)\ \beta^2 - n\ \frac{\gamma^2}{\alpha^2}\ . \tag{56}$$

Therefore, if $\gamma < \beta\alpha$,

$$X_n(1) \to \infty \quad \text{as} \quad n \to \infty\ , \tag{57}$$

and

$$\frac{X_n(2)^{\frac{1}{2}}}{X_n(1)} \to 0 \quad \text{as} \quad n \to \infty\ . \tag{58}$$

That is, the mean of probability density function component $P_n(u)$ tends to ∞, while the component concentrates around its mean. This implies that, for fixed threshold v, the cumulative distribution function component

$$C_n(v) \to 0 \quad \text{as} \quad n \to \infty\ , \tag{59}$$

provided that $\gamma < \beta\alpha$. Thus, expansion (52) has two convergence factors when $\gamma < \beta\alpha$, but otherwise arbitrary β, α, γ; recall (13). Also, if $\gamma < \beta\alpha$, then $r > 0$, and all the terms in component $C_n(v)$ in (53) are positive.

SPECIAL CASE OF $\alpha = 0$

Instead of letting $\alpha \to 0$ in (52) and (53), we return to (23), which gives the probability density function p(u) for $\alpha = 0$. The corresponding cumulative distribution function is

$$C(v) = \int_0^v du\ p(u) = \int_0^{v/\beta} dt\ \frac{t^{\nu-1} \exp(-t)}{\Gamma(\nu)} \sum_{n=0}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n \frac{n!}{(\nu)_n} L_n^{(\nu-1)}(t) =$$

$$= f_o\ \underline{C}\left(\nu,\frac{v}{\beta}\right) + \frac{(v/\beta)^{\nu} \exp(-v/\beta)}{\Gamma(\nu)} \sum_{n=1}^{\infty} f_n \left(\frac{\gamma}{\beta}\right)^n \frac{(n-1)!}{(\nu)_n} L_{n-1}^{(\nu)}\left(\frac{v}{\beta}\right) \quad (60)$$

for $v > 0$, where we used [6; appendix C]. The result in (60) agrees with [6; (91) and (95)]. If desired, we could again let $f_n(\nu,\beta,0,\gamma)\ \gamma^n = \tilde{f}_n(\nu,\beta)$, indicating a two-parameter dependence.

SPECIAL CASE OF $\gamma = 0$

Observe, from (53), that

$$C_n(v) \to \underline{C}\left(\nu+n,\frac{v}{\beta}\right) \quad \text{as } \gamma \to 0\ , \quad (61)$$

meaning that we have, from (52), cumulative distribution function

$$C(v) \to \sum_{n=0}^{\infty} f_n\ \alpha^n\ \underline{C}\left(\nu+n,\frac{v}{\beta}\right) \quad \text{as } \gamma \to 0\ . \quad (62)$$

The corresponding limiting exceedance distribution function, $E(v) = 1 - C(v)$, agrees with [3; (D-21)], when we set $\alpha = 1$, $\beta = 2$, and identify ν, f_n here with K, F g_n there, respectively, and recognize that $\underline{C}(n,x)$ here is $1 - H_{n-1}(x)$ there. Also, since $\underline{C}(\infty,x) = 0$ according to (B-4), the sum in (62) has two convergence factors as $n \to \infty$. Also, $f_n(\nu,\beta,\alpha,0)\ \alpha^n = \tilde{f}_n(\nu,\beta)$.

TYPICAL CHARACTERISTIC FUNCTION COMPONENTS

Up to this point, the given characteristic function $f(\xi)$ has been general; the power series expansion developed in (10) allowed for arbitrary $f(\xi)$ and $\underline{f}(z)$ in (9). Therefore, all the expansions developed in the previous three sections had general coefficients $\{f_n\}$ and $\{g_n\}$ in the expansions; see (38) and (52), for example.

Now, we will consider some typical characteristic functions that have occurred in recent signal processing studies [3,4], and will develop the specific equations for the coefficients $\{f_n\}$. The four types of characteristic function components, that we will concentrate on, are labeled as follows:

$$\text{A:} \quad f(\xi) = (1 - i\xi a)^{-\nu} , \tag{63}$$

$$\text{B:} \quad f(\xi) = \prod_{m=1}^{M} \left(1 - i\xi b_m\right)^{-\nu_m} , \tag{64}$$

$$\text{C:} \quad f(\xi) = \exp\left(i\xi \sum_{m=1}^{M} \frac{\phi_m}{1 - i\xi c_m}\right) , \tag{65}$$

$$\text{D:} \quad f(\xi) = \exp\left(\sum_{m=1}^{M} \frac{\theta_m}{1 - i\xi d_m}\right) . \tag{66}$$

These individual forms may not be valid characteristic functions for some choices of parameter values; that is, the corresponding probability density functions may go negative somewhere. Nevertheless, these functions frequently occur in combinations and are useful as building blocks for more complicated forms.

TYPE A, (63)

The characteristic function of interest is given in (63). The asymptotic decay is in agreement with (8). Function $\underline{f}(z)$ in (9) is then

$$\underline{f}(z) = \left(\frac{\gamma - \beta z}{\gamma - a\alpha - z(\beta - a)}\right)^{\nu} . \tag{67}$$

The product function in (10),

$$F(z) \equiv \left(\frac{\gamma - \beta\alpha}{\gamma - \beta z}\right)^{\nu} \underline{f}(z) = \sum_{n=0}^{\infty} f_n z^n \tag{68}$$

takes the form

$$F(z) = t\ (1 - hz)^{-\nu} , \quad t = \left(\frac{\gamma - \beta\alpha}{\gamma - a\alpha}\right)^{\nu} , \quad h = \frac{\beta - a}{\gamma - a\alpha} , \tag{69}$$

for $\gamma \neq a\alpha$.

Probably, the most efficient way to determine the coefficients $\{f_n\}$ in expansion (68) is to take a logarithm of (69) and expand in a power series:

$$\ln F(z) = \ln t - \nu \ln(1 - hz) = \sum_{n=0}^{\infty} e_n z^n , \tag{70}$$

where the latter coefficients are given by

$$e_o = \ln t , \quad e_n = \frac{\nu}{n} h^n \text{ for } n \geq 1 . \tag{71}$$

Then, exponentiation of (70) immediately leads to coefficients

$$f_o = t , \quad f_n = \frac{\nu}{n} \sum_{k=1}^{n} h^k f_{n-k} \text{ for } n \geq 1 . \tag{72}$$

Here, we used the following useful recursion [6; appendix A]:

$$F(z) = \exp\left(\sum_{n=0}^{\infty} e_n\ z^n\right) = \sum_{n=0}^{\infty} f_n\ z^n \tag{73}$$

according to (70) and (68), with

$$f_o = \exp(e_o)\ , \qquad f_n = \frac{1}{n} \sum_{k=1}^{n} k\ e_k\ f_{n-k} \quad \text{for } n \geq 1\ . \tag{74}$$

Result (72) is an efficient recursion for the desired coefficients $\{f_n\}$. If $\beta = a$, then $t = 1$, $h = 0$, $e_n = 0$ for $n \geq 0$, $f_o = 1$, $f_n = 0$ for $n \geq 1$, and (11) reduces to (63), of course.

TYPE B, (64)

Let $\nu = \nu_1 + \cdots + \nu_M$ in the following. The asymptotic decay of (64) then agrees with (8). Function $\underline{f}(z)$ in (9) is given by

$$\underline{f}(z) = \prod_{m=1}^{M} \left(\frac{\gamma - \beta z}{\gamma - b_m\alpha - z(\beta - b_m)}\right)^{\nu_m}\ , \tag{75}$$

while product function $F(z)$ in (68) is

$$F(z) = t \prod_{m=1}^{M} \left(1 - h_m z\right)^{-\nu_m}\ , \tag{76}$$

where

$$t = \frac{(\gamma - \beta\alpha)^{\nu}}{\prod_{m=1}^{M} (\gamma - b_m\alpha)^{\nu_m}} , \quad h_m = \frac{\beta - b_m}{\gamma - b_m\alpha} \quad \text{for } 1 \le m \le M . \tag{77}$$

In order to develop the power series expansion of (76), observe that

$$\ln F(z) = \ln t - \sum_{m=1}^{M} \nu_m \ln(1 - h_m z) = \sum_{n=0}^{\infty} e_n z^n , \tag{78}$$

with

$$e_o = \ln t , \quad e_n = \frac{1}{n} \mu_n , \quad \mu_n \equiv \sum_{m=1}^{M} \nu_m h_m^n \quad \text{for } n \ge 1 . \tag{79}$$

Then, by use of (73) - (74), we find coefficients

$$f_o = t , \quad f_n = \frac{1}{n} \sum_{k=1}^{n} \mu_k f_{n-k} \quad \text{for } n \ge 1 . \tag{80}$$

A possible initial guess for β follows by matching the $n = 0$ term of expansion (11) to the origin behavior of the given characteristic function; namely, as $\xi \to 0$, (11) gives

$$(1 - i\xi\beta)^{-\nu} \sim 1 + i\xi\beta\nu , \tag{81}$$

while (64) yields

$$f(\xi) \sim 1 + i\xi \sum_{m=1}^{M} b_m \nu_m . \tag{82}$$

Therefore, we could choose initially

$$\beta \cong \frac{1}{\nu} \sum_{m=1}^{M} b_m \nu_m , \quad \nu = \sum_{m=1}^{M} \nu_m . \tag{83}$$

TYPE C, (65)

Here, the asymptotic decay is $\nu = 0$ from (65). There follows

$$F(z) = \underline{f}(z) = \exp\left((\alpha - z) \sum_{m=1}^{M} \frac{q_m}{1 - h_m z}\right) , \tag{84}$$

where

$$q_m = \frac{\phi_m}{\gamma - c_m \alpha} , \quad h_m = \frac{\beta - c_m}{\gamma - c_m \alpha} \quad \text{for } 1 \le m \le M . \tag{85}$$

Then,

$$\ln F(z) = (\alpha - z) \sum_{m=1}^{M} q_m \sum_{n=0}^{\infty} h_m^n z^n = \sum_{n=0}^{\infty} e_n z^n , \tag{86}$$

where

$$e_o = \alpha \sum_{m=1}^{M} q_m , \quad e_n = \sum_{m=1}^{M} q_m h_m^{n-1} (\alpha h_m - 1) \quad \text{for } n \ge 1 . \tag{87}$$

Then, coefficients $\{f_n\}$ follow directly from (74).

Sequence $\{h_m\}$ in (85) can be kept relatively small by choosing $\beta \cong \Sigma c_m /M$. Alternatively, if we set the sum of $\{h_m\}$ to zero, we find $\beta = \Sigma c_m \mu_m /\Sigma \mu_m$, where $\mu_m = 1/(\gamma-\alpha c_m)$. On the other hand, minimizing the sum of $\{h_m^2\}$ gives $\beta = \Sigma c_m \mu_m^2 /\Sigma \mu_m^2$. Since both of these latter two approaches give a β value which depends on γ and α, the first choice seems more reasonable.

A smaller $\{h_m\}$ sequence leads to smaller $\{e_n\}$ and $\{f_n\}$ sequences, in general, thereby promoting faster convergence of expansion (11).

TYPE D, (66)

Again, we have $\nu = 0$. Then, we find

$$F(z) = \underline{f}(z) = \exp\left((\gamma - \beta z) \sum_{m=1}^{M} \frac{q_m}{1 - h_m z}\right) , \tag{88}$$

where

$$q_m = \frac{\theta_m}{\gamma - d_m \alpha} , \quad h_m = \frac{\beta - d_m}{\gamma - d_m \alpha} \quad \text{for } 1 \le m \le M . \tag{89}$$

Then,

$$\ln F(z) = (\gamma - \beta z) \sum_{m=1}^{M} q_m \sum_{n=0}^{\infty} h_m^n z^n = \sum_{n=0}^{\infty} e_n z^n , \tag{90}$$

where

$$e_o = \gamma \sum_{m=1}^{M} q_m , \quad e_n = \sum_{m=1}^{M} q_m h_m^{n-1} (\gamma h_m - \beta) \quad \text{for } n \ge 1 . \tag{91}$$

Finally, sequence $\{f_n\}$ is given by (74) again.

Sequence $\{h_m\}$ in (89) can be kept relatively small by choosing

$$\beta \cong \frac{1}{M} \sum_{m=1}^{M} d_m . \tag{92}$$

This choice is independent of parameters γ and α.

APPLICATION TO FADING

The characteristic function of the output of a multiple-pulse receiver operating in a noisy medium with partially-correlated fading was derived in closed form in [3; (D-1)]. It is

$$f(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left[\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi Q_{mk}\right)\right]^{-\frac{1}{2}} \times$$

$$\times \exp\left[i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{1 - i2\xi Q_{mk}} \right] , \qquad (93)$$

where K is the number of pulses, M is the number of fading components, $\{Q_{mk}\}$ are measures of random signal-to-noise ratios, and $\{e_{mk}\}$ are measures of deterministic signal-to-noise ratios.

This characteristic function in (93) is recognized to be the product of the three types of terms A, B, C in (63), (64), (65), respectively. This means that the logarithms of the individual terms will add. In particular, by reference to (70), it means that the three sets of $\{e_n\}$ coefficients, of each type of characteristic function component, can be added directly together <u>before</u> performing the exponentiation indicated by (73) and (74). We therefore concentrate on obtaining these $\{e_n\}$ coefficients for each type of component A, B, C in (93).

With regard to type A terms, the leading term in (93) leads us to identify $a = 2$, $\nu = -K(\frac{1}{2}M-1)$ in (63). Then, by reference to (69) and (71), we find

$$t = \left(\frac{\gamma - \beta\alpha}{\gamma - 2\alpha}\right)^{-K(\frac{1}{2}M-1)} \quad , \quad h = \frac{\beta - 2}{\gamma - 2\alpha} \; , \tag{94}$$

$$e_o = \ln t \; , \quad e_n = - \frac{K}{n}\left(\frac{M}{2} - 1\right)\left(\frac{\beta - 2}{\gamma - 2\alpha}\right)^n \quad \text{for } n \geq 1 \; . \tag{95}$$

The f_o coefficient follows from (72) as $f_o = t$, as given in (94).

As for type B terms, the second term in (93) requires the identifications $M \to M,K$; $m \to m,k$; $b_m \to 2Q_{mk}$; $\nu_m \to \frac{1}{2}$ in (64). Then, by use of (83), (77), and (79), we find $\nu = MK/2$,

$$t = \frac{(\gamma - \beta\alpha)^{MK/2}}{\left(\prod_{m=1}^{M}\prod_{k=1}^{K}(\gamma - 2\alpha Q_{mk})\right)^{\frac{1}{2}}} \quad , \quad h_m \to \frac{\beta - 2Q_{mk}}{\gamma - 2\alpha Q_{mk}} \; , \tag{96}$$

$$e_o = \ln t \; , \quad e_n = \frac{1}{2n}\sum_{m=1}^{M}\sum_{k=1}^{K}\left(\frac{\beta - 2Q_{mk}}{\gamma - 2\alpha Q_{mk}}\right)^n \quad \text{for } n \geq 1 \; . \tag{97}$$

The f_o coefficient is given by (80) as $f_o = t$, in terms of (96).

Finally, for type C terms, the last term in (93) yields the identifications $M \to M,K$; $m \to m,k$; $\phi_m \to e_{mk}$; $c_m \to 2Q_{mk}$ in (65), in addition to $\nu = 0$. Then, by use of (85) and (87), we find

$$q_m \to \frac{e_{mk}}{\gamma - 2\alpha Q_{mk}} \quad , \quad h_m \to \frac{\beta - 2Q_{mk}}{\gamma - 2\alpha Q_{mk}} \; , \tag{98}$$

$$e_o = \alpha \sum_{m=1}^{M}\sum_{k=1}^{K}\frac{e_{mk}}{\gamma - 2\alpha Q_{mk}} \; , \tag{99}$$

$$e_n = (\beta\alpha - \gamma)\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{(\beta - 2Q_{mk})^{n-1}\, e_{mk}}{(\gamma - 2\alpha Q_{mk})^{n+1}} \quad \text{for } n \geq 1 \; . \tag{100}$$

The f_o coefficient follows from (74) in the form $f_o = \exp(e_o)$, where e_o is given by (99).

When we combine the three types of characteristic function components above, the three individual f_o terms multiply, leading to the composite value

$$f_o = \frac{(\gamma - \beta\alpha)^K \, (\gamma - 2\alpha)^{K(\frac{1}{2}M-1)}}{\left(\prod_{m=1}^{M} \prod_{k=1}^{K} (\gamma - 2\alpha Q_{mk})\right)^{\frac{1}{2}}} \exp\left(\alpha \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{\gamma - 2\alpha Q_{mk}}\right) . \quad (101)$$

Similarly, the three $\{e_n\}$ terms given by (95), (97), and (100) combine (add) to yield composite value

$$n\, e_n = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{(\beta - 2Q_{mk})^{n-1}}{(\gamma - 2\alpha Q_{mk})^{n+1}} \Big((\beta - 2Q_{mk})(\gamma - 2\alpha Q_{mk}) +$$

$$+ 2n\,(\beta\alpha - \gamma)\, e_{mk}\Big) - K\left(\frac{M}{2} - 1\right)\left(\frac{\beta - 2}{\gamma - 2\alpha}\right)^n \quad \text{for } n \ge 1 . \qquad (102)$$

The composite ν value is obtained by adding the three values above, to get $\nu = K(1 - M/2) + MK/2 + 0 = K$.

The coefficients $\{f_n\}$ can be found from (74) in the recursive form

$$f_n = \frac{1}{n} \sum_{k=1}^{n} k\, e_k\, f_{n-k} \quad \text{for } n \ge 1 . \qquad (103)$$

The expansion for the characteristic function is again given by (11), and the corresponding probability density function follows from (19), with the replacement of ν by K in both cases.

SPECIAL CASE OF $\alpha = 0$

In this special case of $\alpha = 0$, there follows $f_o = 1$ from (101), and

$$\gamma^n \, n \, e_n = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} (\beta - 2Q_{mk})^{n-1} (\beta - 2Q_{mk} - 2ne_{mk}) +$$

$$+ K \left(1 - \frac{M}{2}\right) (\beta - 2)^n \quad \text{for } n \geq 1 \, . \tag{104}$$

The characteristic function expansion in (11) now becomes

$$f(\xi) = \frac{1}{(1 - i\xi\beta)^K} \sum_{n=0}^{\infty} f_n \left(\frac{-i\xi\gamma}{1 - i\xi\beta}\right)^n \, . \tag{105}$$

Again, we could write $f_n(\nu,\beta,0,\gamma) \; \gamma^n = \tilde{f}_n(\nu,\beta)$, thereby emphasizing the two-parameter dependence in this special case.

SPECIAL CASE OF $\gamma = 0$

Upon use of $\gamma = 0$ in (101), there follows

$$f_o = \frac{(\beta/2)^K}{\left(\prod_{m=1}^{M} \prod_{k=1}^{K} Q_{mk}\right)^{\frac{1}{2}}} \exp\left(- \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{Q_{mk}}\right) . \tag{106}$$

Also, from (102), we have

$$\alpha^n \, n \, e_n = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 - \frac{\beta}{2Q_{mk}}\right)^{n-1} \left[1 - \frac{\beta}{2Q_{mk}}\left(1 - n \frac{e_{mk}}{Q_{mk}}\right)\right] +$$

$$+ K\left(1 - \frac{M}{2}\right)\left(1 - \frac{\beta}{2}\right)^n \quad \text{for } n \geq 1 . \tag{107}$$

The characteristic function expansion in (11) now becomes

$$f(\xi) = \frac{1}{(1 - i\xi\beta)^K} \sum_{n=0}^{\infty} f_n \left(\frac{\alpha}{1 - i\xi\beta}\right)^n . \tag{108}$$

Here, also, we could write $f_n(\nu,\beta,\alpha,0)\,\alpha^n = \tilde{f}_n(\nu,\beta)$, explicitly indicating the two-parameter dependence in this special case.

As a check on the above results, consider the situation where there is no signal at the system input [3], in which case the parameters in (93) become $e_{mk} = 0$ and $Q_{mk} = 1$. Then, for this noise-only case, (106) yields $f_o = (\beta/2)^K$, and (107) reduces to

$$\alpha^n \, n \, e_n = K\left(1 - \frac{\beta}{2}\right)^n \quad \text{for } n \geq 1 . \tag{109}$$

The corresponding coefficients $\{f_n\}$ can be found in closed form by using (73) directly: let $\zeta = (1 - \beta/2)/\alpha$, getting

$$\sum_{n=0}^{\infty} e_n\, z^n = e_o + K \sum_{n=1}^{\infty} \frac{1}{n}\, \zeta^n\, z^n = e_o - K\, \ln(1 - \zeta z)\ . \qquad (110)$$

Then, performing the exponentiation,

$$F(z) = \exp(e_o)\, (1 - \zeta z)^{-K} = \exp(e_o) \sum_{n=0}^{\infty} \frac{(K)_n}{n!}\, \zeta^n\, z^n\ , \qquad (111)$$

where we used [8; 15.1.8]. There follows the coefficients

$$f_n = f_o\, \frac{(K)_n}{n!}\, \zeta^n = \left(\frac{\beta}{2}\right)^K \frac{(K)_n}{n!} \left(\frac{1 - \beta/2}{\alpha}\right)^n \quad \text{for } n \ge 0\ . \qquad (112)$$

Then, (62), along with $\nu = K$, yields the cumulative distribution function

$$C(v) = \sum_{n=0}^{\infty} f_n\, \alpha^n\, \underline{C}\left(K+n, \frac{v}{\beta}\right) = \left(\frac{\beta}{2}\right)^K \sum_{n=0}^{\infty} \frac{(K)_n}{n!} \left(1 - \frac{\beta}{2}\right)^n \underline{C}\left(K+n, \frac{v}{\beta}\right) \qquad (113)$$

for threshold $v > 0$. Since this result must be independent of β, we can set $\beta = 2$ and get

$$C(v) = \underline{C}\left(K, \frac{v}{2}\right) \quad \text{for } v > 0\ . \qquad (114)$$

(This checks [3; (188) and (D-17)].) The independence of result (113) on β has been numerically verified.

APPLICATION TO NORMALIZATION

The general results derived earlier were specialized to the case where parameter $\gamma = 0$, resulting in expression (62) for the cumulative distribution function. We will further presume that parameter ν is an integer, namely K. Then, a combination of (62), (13), and (B-14) enables us to write the corresponding exceedance distribution function as

$$E(v) = 1 - C(v) = \sum_{n=0}^{\infty} f_n \alpha^n \underline{E}\left(K+n, \frac{v}{\beta}\right) \quad \text{for } v > 0 . \qquad (115)$$

Since $\gamma = 0$, we let $f_n(K,\beta,\alpha,0)\ \alpha^n = \tilde{f}_n(K,\beta)$, emphasizing the two-parameter dependence. This quantity in (115) can be interpreted as the probability that a random variable, denoted **g**, exceeds a <u>fixed</u> threshold v:

$$\Pr(g > v) = E(v) = \sum_{n=0}^{\infty} \tilde{f}_n \underline{E}\left(K+n, \frac{v}{\beta}\right) \quad \text{for } v > 0 . \qquad (116)$$

Expansion coefficients $\{\tilde{f}_n\}$ and parameters K and β are arbitrary in expansion (116).

Now, let us suppose that random variable **g** is not compared with a fixed threshold, but rather with a variable threshold determined from some finite size average of another set of independent random variables. This normalization procedure is adopted in an effort to realize a constant false alarm receiver. In particular, let random variable **h** be evaluated from the average of L noise-only squared envelopes outputs of disjoint narrowband filters. The probability density function of **h** is

then

$$p_o(u) = \frac{u^{L-1} \exp(-\tfrac{1}{2}u/\sigma)}{(2\sigma)^L \ (L-1)!} \quad \text{for } u > 0 \ , \tag{117}$$

where σ is a measure of the noise level in the reference bins.

Random variable **g** is compared with a scaled version of random variable **h**, for purposes of determining whether a signal is present in **g**. This leads us to consider the exceedance (detection) probability of the normalizer in the form

$$\Pr(\mathbf{g} > v\ \mathbf{h}) = \overline{E(v\ \mathbf{h})}^{\mathbf{h}} = \int du\ p_o(u)\ E(v\ u) \ , \tag{118}$$

where v is the scale factor. Here, we used (116) and (117), and the overbar denotes an ensemble average over random variable **h**. When we employ (117), (116), and (B-17), probability (118) can be developed further as follows:

$$\Pr(\mathbf{g} > v\ \mathbf{h}) = \int_0^\infty du\ \frac{u^{L-1} \exp(-\tfrac{1}{2}u/\sigma)}{(2\sigma)^L \ (L-1)!} \sum_{n=0}^{\infty} \tilde{f}_n\ \underline{E}\left(K+n, \frac{v}{\beta}u\right) =$$

$$= \int_0^\infty dx\ \frac{x^{L-1} \exp(-x/2)}{2^L \ (L-1)!} \sum_{n=0}^{\infty} \tilde{f}_n\ \underline{E}\left(K+n, \frac{\sigma v}{\beta}x\right) =$$

$$= \sum_{n=0}^{\infty} \tilde{f}_n \int_0^\infty dx\ \frac{x^{L-1} \exp(-x/2)}{2^L \ (L-1)!} \exp\left(-\frac{\sigma v}{\beta}x\right) \sum_{k=0}^{K-1+n} \frac{1}{k!} \left(\frac{\sigma v}{\beta}x\right)^k =$$

$$= \sum_{n=0}^{\infty} \tilde{f}_n \sum_{k=0}^{K-1+n} \frac{1}{k!} \left(\frac{\sigma v}{\beta}\right)^k \int_0^\infty dx\ \frac{x^{L-1+k}}{2^L \ (L-1)!} \exp\left(-x\left(\frac{1}{2} + \frac{\sigma v}{\beta}\right)\right) =$$

$$= \left(\frac{\beta}{\beta + 2\sigma v}\right)^L \sum_{n=0}^{\infty} \tilde{f}_n \sum_{k=0}^{K-1+n} \binom{L-1+k}{k} \left(\frac{2\sigma v}{\beta + 2\sigma v}\right)^k =$$

$$= \left(\frac{\beta}{\beta + 2\sigma v}\right)^L \sum_{n=0}^{\infty} \tilde{f}_n \; T\left(K-1+n, L-1, \frac{2\sigma v}{\beta + 2\sigma v}\right) . \tag{119}$$

Here, we have defined the auxiliary functions

$$T(m,n,x) \equiv \sum_{k=0}^{m} \binom{n+k}{k} x^k = \sum_{k=0}^{m} (n+1)_k \frac{x^k}{k!} \equiv \sum_{k=0}^{m} U(k,n,x) , \tag{120}$$

which can be easily evaluated by recursions.

Expansion (119) for the normalizer detection probability, Pr(g > v h), is now available for numerical evaluation, once the expansion coefficients $\{f_n\}$ in (115) have been determined. No additional integrals need to be evaluated; rather, only the auxiliary functions in (120) must be evaluated. This procedure parallels that used in [4; pages 13 - 16].

EXTENSION TO ADDITIVE GAUSSIAN NOISE

The expansions derived above are most applicable to the physical situation where filter outputs have been envelope detected, thereby creating positive random variables. If there is also present additive Gaussian noise at the system output, modifications to the expansions must be expected in order to accommodate the negative values that are then possible for the random variable of interest.

Here, we will derive the new expansions for the special case where parameter γ is taken to be zero. We begin by repeating the expansion for the probability density function in (26):

$$p_1(u) = \frac{(u/\beta)^{\nu-1} \exp(-u/\beta)}{\beta\ \Gamma(\nu)} \sum_{n=0}^{\infty} f_n \frac{(\alpha u/\beta)^n}{(\nu)_n} \quad \text{for } u > 0 . \qquad (121)$$

It is presumed that this expansion has already been accomplished for random variable x_1. A second random variable, x_2, which is zero-mean Gaussian, is added to x_1. The probability density function of x_2 is

$$p_2(u) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} \exp\left(-\frac{u^2}{2\sigma^2}\right) \quad \text{for all } u . \qquad (122)$$

The actual derivation of the probability density function of $x = x_1 + x_2$ is accomplished in appendix C by convolving results (121) and (122). The end result is

$$p(u) = \frac{\sigma^{\nu-1} \exp(-\frac{1}{2}u^2/\sigma^2)}{\beta^{\nu}\ \Gamma(\nu)} \sum_{n=0}^{\infty} f_n \frac{(\sigma\alpha/\beta)^n}{(\nu)_n} I_{\nu-1+n}\left(\frac{u}{\sigma} - \frac{\sigma}{\beta}\right) \qquad (123)$$

for all u, where

$$I_\mu(a) \equiv \int_0^\infty dx\ x^\mu\ (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^2 + ax\right) =$$

$$= a\ I_{\mu-1}(a) + (\mu-1)\ I_{\mu-2}(a) \quad \text{for } \mu > 1\ . \tag{124}$$

Starting values for recurrence (124) when μ is integer, that is, when ν in (121) is integer, are given in appendix C. Thus, the simple recurrence in (124) allows for rapid evaluation of the probability density function p(u) in (123), once coefficients $\{f_n\}$ in (121) have been determined.

The cumulative distribution function corresponding to p(u) is also evaluated in appendix C. It is given by

$$C(v) = \sum_{n=0}^{\infty} f_n\ \alpha^n\ J_{\nu-1+n}\left(\frac{v}{\sigma},\frac{\sigma}{\beta}\right) \quad \text{for all } v\ , \tag{125}$$

where

$$J_\mu(a,b) \equiv \int_0^\infty dx\ \frac{x^\mu \exp(-x)}{\Gamma(\mu+1)}\ \Phi\left(a - \frac{x}{b}\right)\ . \tag{126}$$

A recurrence for J_μ is derived in appendix C for the case where μ is integer; in this case, cumulative distribution function C(v) in (125) can be rapidly evaluated once the coefficients $\{f_n\}$ are available. Therefore, the addition of Gaussian noise is easily accommodated, in terms of finding the density and distribution of the sum, with the expansion already available in form (121).

SUMMARY

Expansions of probability density functions and their corresponding cumulative distribution functions have been accomplished in a form that can be interpreted as an extension of the generalized Laguerre expansion. This form does not require or possess any orthogonality properties. Moments or cumulants do not have to be extracted; rather, a linear fractional transformation is made on the available analytical characteristic function, followed by a power series expansion. Applications to a class of characteristic functions typically encountered in signal processing were then made to illustrate the procedure by which the expansion coefficients can be obtained. These results are exact, not approximations; their utility depends on being able to determine the power series expansion mentioned above.

These results have already been used to advantage in numerically evaluating the performance of receivers in fading, both with and without normalization [3,4]. In some cases, appropriate choices of parameter values leads to series of all-positive terms, with their attendant high accuracy, even in the range of very small probabilities [3]. However, it must be noted that these results are very example-dependent and may not extend simply, in general. A new technique and expansion will usually be required for each different problem.

The availability to choose four parameters, ν,β,α,γ, allows for better fits and more rapid convergence to a given

characteristic function or probability density function. This is a worthwhile extension of the Hermite and generalized Laguerre expansions, where only two free parameters are available. Of course, only three of the four parameters are fundamental.

Once the characteristic function expansion for a fixed threshold system has been determined, the probability density and cumulative distribution functions for a related normalizer follow immediately without the need for extensive new derivations. This observation allows for a ready determination of the cost of employing variable detection thresholds, which are inherent in a constant false alarm receiver.

APPENDIX A. FOURIER TRANSFORM OF CUMULATIVE DISTRIBUTION

The cumulative distribution function C(v) is given by

$$C(v) = \int_{-\infty}^{v} du\ p(u) , \tag{A-1}$$

where p(u) is the probability density function, and $f(\xi)$ is the characteristic function of the random variable under consideration. The two-sided Fourier transform of C(v) is

$$g(\xi) = \int dv\ \exp(i\xi v)\ C(v) . \tag{A-2}$$

This latter integral converges at $v = -\infty$ if $\xi_i < a$, where a is a positive constant (and could be $+\infty$, as for a positive random variable). The integral converges at $v = +\infty$ if $\xi_i > 0$. Then, we have

$$g(\xi) = \int dv\ \exp(i\xi v) \int_{-\infty}^{v} du\ p(u) = \int du\ p(u) \int_{u}^{+\infty} dv\ \exp(i\xi v) =$$

$$= \int du\ p(u)\ \frac{\exp(+i\xi\infty) - \exp(i\xi u)}{i\xi} = \frac{-1}{i\xi}\ f(\xi) , \tag{A-3}$$

using $\xi_i > 0$. That is,

$$g(\xi) = \frac{-1}{i\xi}\ f(\xi) \quad \text{for } 0 < \xi_i < a ; \tag{A-4}$$

this relation holds within a strip in the complex ξ-plane.

Let contour D in the complex ξ-plane be the real axis with a slight upward indentation at $\xi = 0$. Then, given the characteristic function $f(\xi)$, we can return to the cumulative

distribution function C(v) by means of

$$\frac{1}{2\pi} \int_D d\xi \, \exp(-iv\xi) \, g(\xi) = C(v) \; . \tag{A-5}$$

EXAMPLE 1: TWO-SIDED EXPONENTIAL

$$p(u) = \frac{1}{2} \exp(-|u|) \quad \text{for all } u \; . \tag{A-6}$$

$$f(\xi) = \frac{1}{1 + \xi^2} \quad \text{for } -1 < \xi_i < 1 \; . \tag{A-7}$$

Here, a = 1. Relation (A-4) yields

$$g(\xi) = \frac{-1}{i\xi(1 + \xi^2)} \quad \text{for } 0 < \xi_i < 1 \; . \tag{A-8}$$

Then, integral (A-5) yields, with contour D passing below $\xi = i$ and above $\xi = 0$, the correct cumulative distribution function

$$\frac{1}{2\pi} \int_D d\xi \, \exp(-iv\xi) \, g(\xi) = \begin{cases} \frac{1}{2} \exp(v) & \text{for } v < 0 \\ 1 - \frac{1}{2} \exp(-v) & \text{for } v > 0 \end{cases} = C(v). \tag{A-9}$$

EXAMPLE 2: ONE-SIDED EXPONENTIAL

$$p(u) = \exp(-u) \quad \text{for } u > 0 \; , \tag{A-10}$$

$$C(v) = \begin{cases} 1 - \exp(-v) & \text{for } v > 0 \\ 0 & \text{for } v < 0 \end{cases} \; , \tag{A-11}$$

$$f(\xi) = (1 - i\xi)^{-1} \quad \text{for } \xi_i > -1 \ . \tag{A-12}$$

Here, $a = +\infty$. From (A-4),

$$g(\xi) = \frac{-1}{i\xi(1 - i\xi)} \quad \text{for } 0 < \xi_i \ . \tag{A-13}$$

Then, (A-5) yields $C(v)$ as given by (A-11), provided that contour D passes above $\xi = 0$.

EXAMPLE 3: GAUSSIAN

$$p(u) = (2\pi)^{-\frac{1}{2}} \exp(-u^2/2) \quad \text{for all } u \ , \tag{A-14}$$

$$C(v) = \Phi(v) \equiv \int_{-\infty}^{v} du \ p(u) \quad \text{for all } v \ , \tag{A-15}$$

$$f(\xi) = \exp(-\xi^2/2) \quad \text{for all } \xi \ . \tag{A-16}$$

Here, $a = +\infty$. Then, from (A-4),

$$g(\xi) = \frac{-1}{i\xi} \exp(-\xi^2/2) \quad \text{for } 0 < \xi_i \ . \tag{A-17}$$

With contour D passing above $\xi = 0$, we find

$$\frac{1}{2\pi} \int_D d\xi \ \exp(-iv\xi) \ g(\xi) = \frac{-1}{2\pi i} \int_D d\xi \ \exp(-i\xi v) \ \frac{\exp(-\xi^2/2)}{\xi} =$$

$$= \frac{-1}{2\pi i}(-i\pi) - \frac{1}{2\pi i} ⨍ d\xi \ \exp(-iv\xi) \ \frac{\exp(-\xi^2/2)}{\xi} =$$

$$= \frac{1}{2} + \frac{1}{2\pi} \int d\xi \ \frac{\sin(v\xi)}{\xi} \exp(-\xi^2/2) =$$

$$= \frac{1}{2} + \left(\Phi(v) - \frac{1}{2}\right) = \Phi(v) \quad \text{for all } v \ . \qquad \text{(A-18)}$$

Here, we used [9; 3.896 4: in particular, integrate both sides with respect to b].

APPENDIX B. INCOMPLETE GAMMA FUNCTION RELATIONS

Define the function

$$\underline{C}(\nu,x) = \int_0^x dy \, \frac{y^{\nu-1} \exp(-y)}{\Gamma(\nu)} \quad \text{for } x \geq 0 \text{ , } \quad \nu > 0 \text{ .} \tag{B-1}$$

(This is $P(\nu,x)$ in [8; 6.5.1].) Then, from [8; 6.5.2 and 6.5.3],

$$\underline{C}(\nu,x) = \frac{\gamma(\nu,x)}{\Gamma(\nu)} = 1 - \frac{\Gamma(\nu,x)}{\Gamma(\nu)} \text{ ,} \tag{B-2}$$

in terms of incomplete gamma functions. The function $\underline{C}(\nu,x)$ is a cumulative distribution function; that is,

$$\underline{C}(\nu,\infty) = 1 \text{ .} \tag{B-3}$$

It is also useful to notice that

$$\underline{C}(\infty,x) = 0 \text{ ,} \tag{B-4}$$

since the unit area of the integrand of (B-1) moves progressively farther out in the positive direction as ν increases. This can be seen from the fact that the density (integrand) in (B-1) has mean ν and standard deviation $\sqrt{\nu}$.

Also, define (density) function

$$p(\nu,x) = \frac{x^{\nu-1} \exp(-x)}{\Gamma(\nu)} \quad \text{for } x \geq 0 \text{ , } \quad \nu > 0 \text{ .} \tag{B-5}$$

Then, function (B-1) satisfies recurrence

$$\underline{C}(\nu,x) = \int_0^x dy\ p(\nu,y) = \underline{C}(\nu-1,x) - p(\nu,x) \quad \text{for } \nu > 1\ , \tag{B-6}$$

while

$$p(\nu,x) = p(\nu-1,x)\ \frac{x}{\nu-1} \quad \text{for } \nu > 1\ . \tag{B-7}$$

Convenient starting values for (B-6) and (B-7), when ν is integer or half-integer, are

$$\underline{C}(\tfrac{1}{2},x) = 2\ \Phi\left((2x)^{\frac{1}{2}}\right) - 1\ , \quad \underline{C}(1,x) = 1 - \exp(-x)\ , \tag{B-8}$$

$$p(\tfrac{1}{2},x) = (\pi x)^{-\frac{1}{2}} \exp(-x)\ , \quad p(1,x) = \exp(-x)\ . \tag{B-9}$$

As an example, with $\nu = n$,

$$p(n,x) = p(n-1,x)\ \frac{x}{n-1} \quad \text{for } n = 2,3,4,\ldots \tag{B-10}$$

$$\underline{C}(n,x) = \underline{C}(n-1,x) - p(n,x) \quad \text{for } n = 2,3,4,\ldots \tag{B-11}$$

along with the starting values for $p(1,x)$ and $\underline{C}(1,x)$ given in (B-9) and (B-8), respectively.

As a second example, with $\nu = n + \frac{1}{2}$,

$$p(n+\tfrac{1}{2},x) = p(n-\tfrac{1}{2},x)\ \frac{x}{n-\frac{1}{2}} \quad \text{for } n = 1,2,3,\ldots \tag{B-12}$$

$$\underline{C}(n+\tfrac{1}{2},x) = \underline{C}(n-\tfrac{1}{2},x) - p(n+\tfrac{1}{2},x) \quad \text{for } n = 1,2,3,\ldots \tag{B-13}$$

along with the starting values for $p(\frac{1}{2},x)$ and $\underline{C}(\frac{1}{2},x)$ given in (B-9) and (B-8), respectively.

We also define exceedance distribution function

$$\underline{E}(\nu,x) = 1 - \underline{C}(\nu,x) = \int_x^\infty dy\, \frac{y^{\nu-1}\exp(-y)}{\Gamma(\nu)} =$$

$$= \Gamma(\nu,x)/\Gamma(\nu) \quad \text{for } x \geq 0 \;, \quad \nu > 0 \;. \tag{B-14}$$

Then,

$$\underline{E}(\nu,x) = \underline{E}(\nu-1,x) + p(\nu,x) \quad \text{for } \nu > 1 \;, \tag{B-15}$$

with recursion (B-7) for $p(\nu,x)$, and starting values

$$\underline{E}(\tfrac{1}{2},x) = 2\;\Phi\left(-(2x)^{\frac{1}{2}}\right) \;, \qquad \underline{E}(1,x) = \exp(-x) \;. \tag{B-16}$$

We also have closed form

$$\underline{E}(n,x) = \exp(-x) \sum_{k=0}^{n-1} \frac{x^k}{k!} \quad \text{for } n \geq 1 \;. \tag{B-17}$$

APPENDIX C. RECURRENCES FOR SUM OF RANDOM VARIABLES

The probability density functions of random variables x_1 and x_2 were given in (121) and (122), respectively. We will now derive the probability density function of sum $x = x_1 + x_2$, by convolving these two functions. We have

$$p(u) = \int dt\; p_1(t)\; p_2(u-t) =$$

$$= \frac{(2\pi)^{-\frac{1}{2}}}{\beta^{\nu}\;\Gamma(\nu)\;\sigma} \sum_{n=0}^{\infty} f_n \frac{(\alpha/\beta)^n}{(\nu)_n} \int_0^{\infty} dt\; t^{\nu-1+n} \exp\left(-\frac{t}{\beta} - \frac{(u-t)^2}{2\sigma^2}\right) =$$

$$= \frac{\sigma^{\nu-1}\exp(-\frac{1}{2}u^2/\sigma^2)}{\beta^{\nu}\;\Gamma(\nu)} \sum_{n=0}^{\infty} f_n \frac{(\sigma\alpha/\beta)^n}{(\nu)_n} I_{\nu-1+n}\left(\frac{u}{\sigma} - \frac{\sigma}{\beta}\right) \text{ for all } u, \quad \text{(C-1)}$$

where $I_{\mu}(a)$ was defined in (124). If ν is integer, the recurrence may be started with

$$I_o(a) = \exp(a^2/2)\;\Phi(a)\;, \qquad I_1(a) = a\;I_o(a) + (2\pi)^{-\frac{1}{2}}\;, \qquad \text{(C-2)}$$

where normalized Gaussian cumulative distribution function

$$\Phi(a) \equiv \int_{-\infty}^{a} dx\;\phi(x)\;, \qquad \phi(x) \equiv (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)\;. \qquad \text{(C-3)}$$

The cumulative distribution function corresponding to p(u) is

$$C(v) = \int_{-\infty}^{v} du\; p(u) = \int_0^{\infty} du\; p_1(u)\; C_2(v-u) =$$

$$= \sum_{n=0}^{\infty} f_n\;\alpha^n \int_0^{\infty} du\; \frac{u^{\nu-1+n}\exp(-u/\beta)}{\beta^{\nu+n}\;\Gamma(\nu+n)}\;\Phi\left(\frac{v-u}{\sigma}\right) =$$

$$= \sum_{n=0}^{\infty} f_n \, \alpha^n \, J_{\nu-1+n}\left(\frac{v}{\sigma}, \frac{\sigma}{\beta}\right) \quad \text{for all } v \ , \tag{C-4}$$

where $J_\mu(a,b)$ was defined in (126).

A recurrence for J_μ is now derived for the case where μ is an integer m; we have, for $b > 0$,

$$J_m(a,b) = \int_0^\infty dx \, \frac{x^m \exp(-x)}{m!} \, \Phi\left(a - \frac{x}{b}\right) = \int_0^\infty dx \, \frac{x^m \exp(-x)}{m!} \int_{-\infty}^{a-x/b} dt \, \phi(t) =$$

$$= \int_{-\infty}^{a} dt \, \phi(t) \int_0^{b(a-t)} dx \, \frac{x^m \exp(-x)}{m!} = \int_{-\infty}^{a} dt \, \phi(t) \left(1 - \sum_{k=0}^{m} \frac{b^k (a-t)^k}{k!} \, e^{b(t-a)}\right)$$

$$= \Phi(a) - \sum_{k=0}^{m} \frac{b^k}{k!} \int_{-\infty}^{a} dt \, \phi(t) \, (a-t)^k \exp(b(t-a)) =$$

$$= \Phi(a) - \sum_{k=0}^{m} \frac{b^k}{k!} \int_0^\infty dx \, (2\pi)^{-\frac{1}{2}} \, x^k \exp\left(-\frac{1}{2}(a-x)^2 - bx\right) =$$

$$= \Phi(a) - \exp(-a^2/2) \sum_{k=0}^{m} \frac{b^k}{k!} \, I_k(a - b) \ . \tag{C-5}$$

There follows immediately the simple recursion

$$J_m(a,b) = J_{m-1}(a,b) - \exp(-a^2/2) \, \frac{b^m}{m!} \, I_m(a - b) \quad \text{for } m \geq 1 \ . \tag{C-6}$$

This can be started with the value

$$J_o(a,b) = \Phi(a) - \exp\left(\frac{1}{2}b^2 - ab\right) \Phi(a - b) \ . \tag{C-7}$$

Since probability density function $x^{\mu} \exp(-x)/\Gamma(\mu+1)$ in (126) has mean $\mu+1$ and standard deviation $(\mu+1)^{\frac{1}{2}}$, its area moves to the right on the x-axis as μ increases. Therefore, the $\Phi(a - x/b)$ function in (126) forces $J_{\mu}(a,b) \to 0$ as $\mu \to \infty$. This observation implies that the cumulative distribution function expansion in (125) has two convergence factors, namely $f_n \alpha^n$ and $J_{\nu-1+n}$.

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